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(Research Article)

Fourier Sine Transform Method for the Free Vibration of Euler-Bernoulli Beam Resting on Winkler Foundation

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Abstract

In this study, the fourth order homogeneous partial differential equation (PDE) governing the free vibrations of Euler-Bernoulli beams on Winkler foundation with prismatic cross-sections was solved using the finite Fourier sine integral transformation method. Euler-Bernoulli beam theory was used to model the beam while Winkler foundation model was used for the foundation. The beam of length l was assumed to be simply supported at the ends $x = 0$, and $x = l$. The PDE was decoupled by the assumption of harmonic vibration. Application of the finite Fourier sine integral transformation on the decoupled equation resulted in the transformation of the problem to an algebraic eigenvalue problem. The condition for non-trivial solutions resulted to the characteristic frequency equation which was expressed in terms of a non-dimensional frequency parameter $\overline{\omega}_n$. The frequency equation which was observed to be the exact frequency equation obtained in the literature using the Navier series method, was solved to obtain the non-dimensional frequencies. Numerical values of the non-dimensional frequencies were computed for the case where $4\beta^4 = 1$, $l = 1$, and for $n = 1, 2, 3, 4, 5$. It was found that exact values of the non-dimensional frequencies were obtained using the present method.

Keywords: Fourier sine transform method, Euler-Bernoulli beam, Winkler foundation, Natural frequencies, Characteristic (frequency) equation.

1. Introduction

The problem of beams resting on elastic foundations is commonly encountered in the analysis and design of the foundations of buildings, highways and airport runways, railways, retaining walls and of geotechnical structures in general [1-4]. In the mathematical formulation of the problems of beams resting on elastic foundations, the beam theories that have been used include: the Euler-Bernoulli theory [5], Timoshenko theory [4], and refined/shear deformable theories [6]. Models of elastic foundations that have been used are: the Winkler one parameter model [7, 8], the two parameters models (Borodich-Filonenko, Vlasov, Hetenyi, and Pasternak) [9, 10, 11], three parameter models [12], multi-parameter models and the elastic continuum models.

Winkler foundation model is the simplest and most frequently used model. The model assumes that the subgrade/foundation reaction at any point is directly proportional to the beam deflection at the point. These results in a model that has discontinuity in deflection at the unloaded parts of the soil, in violation of the results of the theory of elasticity. However, despite the shortcoming, the Winkler model has been commonly used due to the mathematical simplicity of the resulting formulation [13, 14]. This paper is focused on the Euler-Bernoulli beam theory to model the beam and the Winkler model for the foundation. The fundamental assumptions of the Euler-Bernoulli beam theory used in this work are:

- The beam is long relative to its depth and width.
- Stresses perpendicular to the beam length are much smaller than those parallel and can be ignored.
- The beam cross-section is constant along its longitudinal axis and varies smoothly.
- The beam cross-section has a longitudinal plane of symmetry. The resultant of the transverse loads acting on each section lies on that plane. The support conditions are also symmetric about this plane.
- Transverse deflections and rotations of the beam are so small that the small deflection equations of strain-displacement apply.
- Beam material is isotropic and linear elastic.
- Plane sections originally normal to the longitudinal axis remain plane and normal to the deformed

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longitudinal axis after bending. This is called the normality condition.

Dynamic analysis of Euler-Bernoulli beam on Winkler foundation problems is an important aspect of structural and geotechnical investigation, and the natural frequencies of free vibration obtained are useful in that context. The determination of the natural frequency of free vibration and the associated mode shapes is important in vibration isolation problems. The natural vibration analysis of beams resting on elastic foundations have been extensively studied by many scholars; including Raftoyiannisi et al [15], Coskun et al [16], Öztürk et al [17], Öztürk and Coskun [18], Lai et al [19], Thambiratnam and Zhuge [20], Blevins [21] and Mohanta et al [22].

Lai et al [19] used the finite element method based on the exact displacement shape functions to derive solutions for the free vibrations of uniform prismatic beams resting on uniform elastic foundations. De Rosa [4] determined the natural vibration frequencies of Timoshenko beams resting on two-parameter elastic foundations. Matsunuga [9], assumed power series expansion of the displacement components and derived a set of equations of one-dimensional higher order theory of deep elastic beam-columns resting on elastic foundations by using the Hamilton's principle. Chen [5] studied the vibration of a beam resting on an elastic foundation by using the differential quadrature element method (DQEM). Balkaya et al [3] determined the natural frequencies of a pipeline modeled as a uniform beam resting on a Winkler and Pasternak soil by using the differential transform method. The methods that have been used to solve the dynamic beam on elastic foundation problem are the: Homotopy perturbation method, variational iteration method, differential quadrature element method (DQEM), differential transform method (DTM) and differential quadrature method (DQM).

2. Theoretical Framework / Governing Equation

The governing differential equation of equilibrium of Euler-Bernoulli beams resting on Winkler foundation is given by:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + kw(x,t) + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = p(x,t) \qquad (1)$$

For $0 \le x \le l$

where E is the Young's modulus of elasticity of the beam, I is the area moment of inertia of the beam about the neutral axis, ρ is the mass density of the beam, A is the cross-sectional area of the beam, k is the modulus of the Winkler foundation, $p(x, t)$ is the externally applied excitation force, $w(x, t)$ is the time dependent deflection function, t is the time variable, x is the longitudinal axis of the beam and l is the span of the beam.

The problem considered in this study is the Euler-Bernoulli beam with prismatic cross-section resting on Winkler foundation where the beam is simply supported at the ends $x = 0$, and $x = l$. The natural frequencies of vibration of the beam are to be determined using the Fourier series integral transform method.

3. Methodology

The Fourier sine transform method is an integral transformation technique that employs a sinusoidal kernel, and the method has been found to be ideally suited to boundary value problems with Dirichlet boundary conditions. Due to the simply supported ends of the Euler-Bernoulli beam at $x = 0$, and $x = l$, the boundary conditions are of the Dirichlet type, making the Fourier sine transform method ideal as a mathematical tool for the research. The disadvantage of the Fourier sine transform method is the difficulty in dealing with boundary value problems with non – Dirichlet or Neumann boundary conditions. In such cases, the Fourier sine kernel may be combined with polynomials which are constructed to satisfy aproiri the boundary conditions. Alternatively, the Fourier cosine transform method may be used if the ends are clamped. The simplicity of the Fourier sine transform method in solving boundary value problems with the Dirichlet type boundary conditions is a powerful attraction of the method.

Another advantage of the method is its ability to obtain closed from mathematical solutions to the boundary value problem.

For free (natural) vibrations, the excitation force,

$$p(x,t) = 0 \qquad (2)$$

The governing differential equation of equilibrium reduces to the homogeneous equation:

$$EI\frac{\partial^4 w}{\partial x^4} + kw + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \qquad (3)$$

For harmonic vibrations,

$$w(x,t) = W(x)e^{i\omega_n t} \qquad (4)$$

Then, the governing equation becomes:

$$\frac{\partial^4 W}{\partial x^4}e^{i\omega_n t} + \frac{k}{EI}W(x)e^{i\omega_n t} - \frac{\rho A w_n^2}{EI}W(x)e^{i\omega_n t} = 0 \qquad (5)$$

$$\left(W^{iv}(x) + 4\beta^4 W(x) - \frac{\rho A w_n^2}{EI}W(x)\right)e^{i\omega_n t} = 0 \qquad (6)$$

Where the primes denote derivatives with respect to *x*.

$$4\beta^4 = \frac{k}{EI} \tag{7}$$

For non-trivial solutions,

$$W^{iv}(x) + \left(4\beta^4 - \frac{\rho A\omega_n^2}{EI}\right)W(x) = 0 \tag{8}$$

$$\text{Let } \frac{\rho A\omega_n^2}{EI} = \bar{\omega}_n^2 \cdot 4\beta^4 \tag{9}$$

Then,

$$W^{iv}(x) + 4\beta^4(1-\bar{\omega}_n^2)W(x) = 0 \tag{10}$$

We apply the finite Fourier sine transformation to obtain

$$\int_0^l \left(W^{iv}(x) + 4\beta^4(1-\bar{\omega}_n^2)W(x)\right)\sin\frac{n\pi x}{l}\,dx = 0 \tag{11}$$

4. Results

The linearity property of the Fourier sine transformation is used to obtain:

$$\int_0^l W^{iv}(x)\sin\frac{n\pi x}{l}\,dx + 4\beta^4(1-\bar{\omega}_n^2)\int_0^l W(x)\sin\frac{n\pi x}{l}\,dx = 0 \tag{12}$$

Further simplification yields:

$$\left(\frac{n\pi}{l}\right)^4 \int_0^l W(x)\sin\frac{n\pi x}{l}\,dx + 4\beta^4(1-\bar{\omega}_n^2)\int_0^l W(x)\sin\frac{n\pi x}{l}\,dx = 0 \tag{13}$$

Let

$$\int_0^l W(x)\sin\frac{n\pi x}{l}\,dx = W_n \tag{14}$$

where W_n is the Fourier transform of $w(x)$

Then Equation (13) could be expressed as an algebraic problem, thus:

$$\left(\frac{n\pi}{l}\right)^4 W_n + 4\beta^4(1-\bar{\omega}_n^2)W_n = 0 \tag{15}$$

Simplifying, we obtain the eigenvalue-eigenvector problem:

$$\left(\left(\frac{n\pi}{l}\right)^4 + 4\beta^4(1-\bar{\omega}_n^2)\right)W_n = 0 \tag{16}$$

For non-trivial solutions,

$$W_n \neq 0 \tag{17}$$

Then, the characteristic (frequency) equation become:

$$\left(\frac{n\pi}{l}\right)^4 + 4\beta^4(1-\bar{\omega}_n^2) = 0 \tag{18}$$

Solving,

$$\bar{\omega}_n^2 = \frac{\left(\frac{n\pi}{l}\right)^4 + 4\beta^4}{4\beta^4} = \frac{\rho A\omega_n^2}{EI} \tag{19}$$

$$\bar{\omega}_n = \left(1 + \frac{1}{4\beta^4}\left(\frac{n\pi}{l}\right)^4\right)^{1/2} \tag{20}$$

$$\text{Let } 4\beta^4 = 1 \tag{21}$$

$$l = 1 \tag{22}$$

The natural frequencies become:

$$\bar{\omega}_1 = (1+\pi^4)^{1/2} = 9.920135636 \tag{23}$$

$$\bar{\omega}_2 = \left(1 + \left(\frac{2\pi}{l}\right)^4\right)^{1/2} = \left(1+(2\pi)^4\right)^{1/2} = 39.49108072 \tag{24}$$

$$\bar{\omega}_3 = (1+(3\pi)^4)^{1/2} = 88.83206839 \tag{25}$$

$$\bar{\omega}_4 = (1+(4\pi)^4)^{1/2} = 157.9168367 \tag{26}$$

$$\bar{\omega}_5 = (1+(5\pi)^4)^{1/2} = 246.7421364 \tag{27}$$

The numerical solutions agree with the solutions obtained by Chen [5] who solved the same problem using the differential transform method (DTM).

The same exact solution can be obtained by using the method of Fourier series. In the method of Fourier series, the dynamic deflection function $w(x, t)$ is assumed for harmonic vibrations as a Fourier sine series of the form

$$w(x,t) = \sum_{n=1}^{\infty} W_n \sin\frac{n\pi x}{l}\exp(i\omega_n t) \tag{28}$$

where W_n is the Fourier coefficient of $w(x)$. $n = 1, 2, 3, \ldots \infty$. We observe that $w(x, t)$ satisfies the Dirichlet boundary conditions since $w(x = 0, t) = 0$ and $w''(x = 0, t) = 0$.

Then, the governing equation becomes

$$EI\frac{\partial^4}{\partial x^4}\sum_{n=1}^{\infty}W_n \sin\frac{n\pi x}{l}e^{i\omega_n t}+k\sum_{n=1}^{\infty}W_n \sin\frac{n\pi x}{l}e^{i\omega_n t} + \rho A\frac{\partial^2}{\partial t^2}\sum_{n=1}^{\infty}W_n \sin\frac{n\pi x}{l}e^{i\omega_n t}=0 \quad (29)$$

Simplification yields:

$$\sum_{n=1}^{\infty}\left\{EI\left(\frac{n\pi}{l}\right)^4+\left(k-\rho A\omega_n^2\right)\right\}W_n \sin\frac{n\pi x}{l}e^{i\omega_n t}=0 \quad (30)$$

This becomes an eigenvalue – eigenvector problem.

For non-trivial solutions to Equation (31)

$$\left\{EI\left(\frac{n\pi}{l}\right)^4+\left(k-\rho A\omega_n^2\right)\right\}W_n=0 \quad (31)$$

we have: $W \neq 0$. Then,

$$EI\left(\frac{n\pi}{l}\right)^4+k-\rho A\omega_n^2=0 \quad (32)$$

$$\left(\frac{n\pi}{l}\right)^4+\frac{k}{EI}-\frac{\rho A\omega_n^2}{EI}=0 \quad (33)$$

$$\left(\frac{n\pi}{l}\right)^4+4\beta^4-\frac{\rho A\omega_n^2}{EI}=0 \quad (34)$$

$$\left(\frac{n\pi}{l}\right)^4+4\beta^4\left(1-\frac{\rho A\omega_n^2}{4\beta^4 EI}\right)=0 \quad (35)$$

$$\left(\frac{n\pi}{l}\right)^4+4\beta^4\left(1-\overline{\omega}_n^2\right)=0 \quad (36)$$

Where $\overline{\omega}_n^2=\frac{\rho A\omega_n^2}{4\beta^4 EI}=\frac{\rho A}{k}\omega_n^2$ (37)

Solving, we obtain an identical equation obtained earlier as Equation (20) using the Fourier sine transform method:

$$\overline{\omega}_n=\left(1+\frac{1}{4\beta^4}\left(\frac{n\pi}{l}\right)^4\right)^{1/2} \quad (38)$$

For $4\beta^4=1$, and $l=1$ identical results are obtained for $\overline{\omega}_n$ for n = 1, 2, 3, 4, 5, …

5. Discussion

The finite Fourier sine transformation has been successfully used in this work to solve the differential equation of equilibrium governing the free vibrations of Euler-Bernoulli beams with prismatic cross-sections. The governing equation solved was Equation (3), a homogeneous partial differential equation with the dynamic deflection function $w(x, t)$, as the unknown independent variable. The assumption of harmonic vibrations meant the dynamic deflection function could be expressible in variable-separable form as Equation (4). This led to a decoupling of the partial differential equation to obtain Equation (6). The conditions for nontrivial solution resulted in the ordinary differential equation in terms of $W(x)$ given by Equation (8) or Equation (10). The finite Fourier sine transformation was then applied to Equation (10) to obtain Equation (11). The linearity property of the finite Fourier sine transformation was used to express the transformed equation as Equation (12). Simplification of the transformed equation resulted to Equations (13) and (16). The transformed Equation (16) was observed to be an algebraic eigenvalue problem. The conditions for nontrivial solutions resulted in the characteristic (frequency) equation given by Equation (18), which was solved to obtain the dimensionless frequency parameter $\overline{\omega}_n$ as Equation (20). Numerical values of the dimensionless frequency parameter $\overline{\omega}_n$ were obtained for $4\beta^4=1$, and l = 1 and calculated for n = 1, 2, 3, 4, 5. The calculated values were given as Equations (23) – (27). It was observed that the calculated values were exactly the same as the exact solutions for the same problem found in the literature.

6. Conclusions

From the study, the following conclusions are made:

- The finite Fourier sine integral transformation being a linear transformation is ideal for the solution of linear partial differential equations of fourth order which governs the free harmonic vibration of Euler-Bernoulli beam of prismatic cross-section.
- The beam's simply supported ends x = 0, and $x = l$ satisfy the Dirichlet boundary conditions and offer simplifications to the evaluation of the finite Fourier sine transforms of the derivatives of the unknown independent variable $W(x)$.
- The application of the finite Fourier sine transformation simplifies the problem of free harmonic vibration of simply supported Euler-Bernoulli beams with prismatic cross-section from a differential equation to an algebraic eigenvalue equation.
- The characteristic frequency equations obtained for the free harmonic vibration of prismatic Euler-Bernoulli beams with simply supported ends (x = 0, and $x = l$) are the exact frequency equations obtained by other researchers in the literature, using Navier's series and energy minimization methods.

- The frequency obtained for the case considered where $4\beta^4 = 1$, l = 1 and n = 1, 2, 3, 4, 5 … are the exact frequencies obtained in literature by other scholars.
- The solutions obtained by the finite Fourier sine transform method has been validated by the classical Fourier sine series method, which gave identical results for the problem considered.
- The Fourier sine transform method yielded closed from matheatical solutions for the natural frequencies of vibration of Euler-Bernoulli beam resting on Winkler foundation.

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